

A new separation theorem with geometric applications

Farhad Shahrokhi

Department of Computer Science and Engineering, UNT
P.O.Box 13886, Denton, TX 76203-3886, USA farhad@cs.unt.edu

Abstract

Let $G = (V(G), E(G))$ be an undirected graph with a measure function μ assigning non-negative values to subgraphs H so that $\mu(H)$ does not exceed the clique cover number of H . When μ satisfies some additional natural conditions, we study the problem of separating G into two subgraphs, each with a measure of at most $2\mu(G)/3$ by removing a set of vertices that can be covered with a small number of cliques G . When $E(G) = E(G_1) \cap E(G_2)$, where $G_1 = (V(G_1), E(G_1))$ is a graph with $V(G_1) = V(G)$, and $G_2 = (V(G_2), E(G_2))$ is a chordal graph with $V(G_2) = V(G)$, we prove that there is a separator S that can be covered with $O(\sqrt{l(G, G_1)})$ cliques in G , where $l = l(G, G_1)$ is a parameter similar to the bandwidth, which arises from the linear orderings of cliques covers in G_1 . The results and the methods are then used to obtain exact and approximate algorithms which significantly improve some of the past results for several well known NP-hard geometric problems. In addition, the methods involve introducing new concepts and hence may be of an independent interest.

1 Introduction and Summary

Separation theorems have shown to play a key role in the design of the divide and conquer algorithms, as well as solving extremal problems in combinatorial topology and geometry. The earliest result in this area is a result of Lipton and Tarjan [10] that asserts any n vertex planar graph can be separated into two subgraphs with at most $\frac{2n}{3}$ vertices by removing only $O(\sqrt{n})$ vertices. This result is extended by many authors including Miller et al [11], Fox and Pach [6], [7], and Chan [3].

Clearly if a graph contain a large clique, then it can not have a separation property that resembles the planar case. Fox and Pach [6, 7] have recently studied the string graphs which contain the class of planar graphs, and have shown that when these graphs do not contain a $K_{t,t}$, of fixed size t , as a subgraph, then a suitable separator exists. Although this powerful result is extremely effective in solving extremal problems, its computational power is limited to graphs that do not contain a "large" complete bipartite subgraph. Chan [3] studied the problem of computing the packing and

piercing numbers of fat objects in R^d , where the dimension d is fixed. He drastically improved the running time of the first polynomial time approximation scheme (PTAS) for packing of fat objects due to Erlebach et al [5], and also provided the first PTAS for the piercing problem of fat objects. Parts of Chan's [3] work involved proving a separation theorem with respect to the abstract concept of a measure on fat objects. Motivated by his work we have defined the notation of a measure in a more combinatorial fashion on graphs. Furthermore, we have proven a combinatorial separation theorem. It should however be noted that the results in [3] do not imply ours, and our results do not apply to the general fat objects.

Let μ be a function that assigns non-negative values to subgraphs of G . μ is called a measure function if the following hold.

- (i) $\mu(H_1) \leq \mu(H_2)$, if $H_1 \subseteq H_2 \subseteq G$, (ii) $\mu(H_1 \cup H_2) \leq \mu(H_1) + \mu(H_2)$, if $H_1, H_2 \subseteq G$, (iii) $\mu(H_1 \cup H_2) = \mu(H_1) + \mu(H_2)$, if there are no edges between H_1 and H_2 .

Central to our result is a length concept similar to the bandwidth. Let H be a graph with $V(G) = V(H)$ and $E(G) \subseteq E(H)$ and let $C = \{C_1, C_2, \dots, C_k\}$ be a clique cover in H . For any $e = xy \in E(G)$, $x \in C_l, y \in C_t$, define $l(e, G, H, C)$ to be $|t - l|$. Let $l(G, H, C)$ denote $\max_{e \in E(G)} l(e, G, H, C)$, and let $l(G, H)$ denote $\min_{C \in \mathcal{C}} l(G, H, C)$, where \mathcal{C} denotes the set of all ordered clique covers in H . We refer to $l(G, H)$ as the *length* of G in H . It is important to note that when $l(G, H)$ is small, then G exhibits some nice separation properties. For instance, one can partition $V(G)$ into blocks of $l(G, H)$ consecutive cliques of H , and argue that removal of any block separates G . Particularly, when $l(G, H) = 1$, then one can separate G by removing one clique from H . Similar important concepts such as treewidth, pathwidth, and bandwidth have been introduced in the past [2], but none is identical to the concept of length introduced here. Clearly $l(G, G) \leq BW(G)$, where $BW(G)$ is the bandwidth of G . Moreover as we will see, there is a simple but important connection between $L(G, G)$ and the dimension of interval orders.

Recall that a chordal graph does not have a chordless cycle of length at least 4. Our main result which

is Theorem 1 is a generalization of the result stated earlier in the abstract, to $p \geq 2$ graphs, where G_p is a chordal graph.

Theorem 1 *Let μ be a measure on $G = (V(G), E(G))$, and let G_1, G_2, \dots, G_p be graphs with $V(G_1) = V(G_2) = \dots, V(G_p) = V(G)$, $p \geq 2$ and $E(G) = \cap_{i=1}^p E(G_i)$ so that G_p is chordal. Then there is a vertex separator S in G whose removal separates G into two subgraph so that each subgraph has a measure of at most $2\mu(G)/3$. In addition, the induced graph of G on S can be covered with at most $O(2^{p l^* \frac{p-1}{p}} \mu(G)^{\frac{p-1}{p}})$ many cliques from G , where $l^* = \max_{1 \leq i \leq p-1} l(G, G_i)$.*

Proof of Theorem 1 combines the clique separation properties of chordal graphs and perfect elimination trees, together with the properties associated with the length of a graph. The theorem either finds a suitable clique separator in the chordal graph G_p , or identifies a graph G_j , for which the cardinality of the clique cover is large, and the separates G using length properties. An effective application of Theorem 1 to a specific problem normally requires to define μ so that, $\mu(G) = O(|C|)$, for an appropriate clique cover C in G .

The time complexity of finding the separator depends on the structure of the measure and how fast we can compute the measure on any subgraph. In typical applications of interest with $p = 2$, the separation algorithm can be implemented to run better than $O(|V(G)|^2)$.

2 Applications

Proper applications of Theorem 1 gives rise to the following.

Theorem 2 *Let $G = (V(G), E(G))$ be the intersection graph of a set of axis parallel unit height rectangles in the plane. Then, a maximum independent set in G can be computed in $|V(G)|^{O(\sqrt{\alpha(G)})}$, where $\alpha(G)$ is the independence number of G . Moreover, there is a PTAS that gives a $(1 - \epsilon)$ -approximate solution to $\alpha(G)$ in $|V(G)|^{O(\frac{1}{\epsilon})}$ time and requires $O(|V(G)|)^2$ storage.*

Proof sketch. For $R_1, R_2 \in V(G)$, define $R_1 \prec_1 R_2$, if there is a horizontal line L so that R_1 is above L and R_2 is below L . Likewise, define $R_1 \prec_2 R_2$, if there is a vertical line L so that R_1 is to the left of L and R_2 is to the right of L . Observe that G_i , the incomparability graphs for \prec_i is an interval graph, $i = 1, 2$, and hence is chordal. It is further easy to verify that $l(G, G_1) = 1$. Finally, let C be a 2-approximate solution for the clique cover number

of G that also provides for a $1/2$ -approximate solution to independence number of G , and for any induced subgraph F , define $\mu(F)$ to be the restriction of C to F . (Note that $\mu(G)$ can be computed in $O(|V(G)| \log(|V(G)|))$ time [4].)

For obtaining the sub-exponential time algorithm one can adopt the method in [10] proposed for planar graphs, by enumerating independent sets inside of separator and then recursively applying Theorem 1 to G . For the PTAS, one can also use the original approach in [10] adopted by Chan [3], by recursively separating G , but terminating the recursion when for a subgraph F , $\mu(F) = O(\frac{1}{\epsilon})$ and then applying the sub-exponential algorithm to F . \square

Similarly, one can prove the following.

Theorem 3 *Let S be a set of axis parallel unit height rectangles in the plane. Then, the piercing number of S can be computed in $|S|^{O(\sqrt{P(S)})}$, where $P(S)$ is the piercing number of S . Moreover, there is a PTAS that gives a $(1 + \epsilon)$ -approximate solution to $P(S)$ in $|S|^{O(\frac{1}{\epsilon})}$ time and requires $O(|S|)^2$ storage.*

Our sub-exponential time algorithms in Theorems 2 and 3 are the first ones for the unit height rectangles, and we are not aware of any previous sub-exponential algorithms for these problem. Moreover, the storage requirement for the PTAS in Theorem 2 drastically improves upon $|V(G)|^{O(\frac{1}{\epsilon})}$ storage requirement of the best known previous algorithm in [1], due to Agarwal et al, that was combining dynamic programming with the shifting method. Finally the time complexity of PTAS in Theorem 3 drastically improves upon $|S|^{O(\frac{1}{\epsilon})}$ in [4].

Theorem 4 *Let $P, |P| = n$ be a set of points in the plane. There is an algorithm for computing the minimum number of discs of unit diameter needed to cover all points in P that gives an answer in $n^{O(\sqrt{\text{opt}(P)})}$ time, and $O(n^2)$ storage where $\text{opt}(P)$ is the value of an optimal solution. Furthermore, there is a PTAS that gives a $(1 + \epsilon)$ -approximate solution in $n^{O(\frac{1}{\epsilon})}$ time, and $O(n^2)$ storage.*

Proof sketch. For graph G , let $V(G) = P$, and for any $x, y \in P$, if they of distance at most 1, then place $xy \in E(G)$. Next, as suggested in [9] consider a square n by n grid in the plane containing all the points, so that each cell in the grid is a unit square. Note that the grid can be placed so that no two points appear in the boundary of a cell. Define two interval orders \prec_1 and \prec_2 on $V(G)$ as follows. $x \prec_1 y$, if $xy \notin E(G)$ and x and y are in different vertical strips of the grid so that x is to the left of

y . $x \prec_2 y$, if $xy \notin E(G)$ and there is horizontal line L in the plane so that x is above L and y is below L . Let G_i , $i = 1, 2$ be the incomparability interval graph associated with \prec_i , and note that points in any vertical strip of the grid constitute a clique of G_1 and hence $l(G, G_1) = 1$.

For any $xy \in E(G)$, $x, y \in E(G)$, place two discs in the plane that has x and y in its boundary and call the resulting multi-set of discs \mathcal{C} , and note that $|\mathcal{C}| = O(n^2)$. If G is disconnected, then we would solve the problem for each component, and take the union of the solutions, so we will assume that G is connected. Thus, we can assume with no loss of generality that any feasible solution C' for any $P' \subseteq P$ is a subset of \mathcal{C} , for otherwise we can replace any $D \in C'$ by one disc from \mathcal{C} . Furthermore, it is easy to construct a feasible solution C so that $|C| \leq c\beta(G)$, where $\beta(G)$ is the clique cover number of G and c is a constant no more than 16, in about $O(n^2)$ time. Thus $\beta(G) \leq \text{opt}(P) \leq |C| \leq 16\beta(G)$. For any induced subgraph F in G , define $\mu(F)$ to be $|C_F|$, or, the cardinality of the restriction of C to F , and note that Theorem 1 applies.

Finally follow the details in [10] and the previous theorems by noting that we will always select our disc cover solutions from \mathcal{C} . (Note that the time complexity of enumeration inside of the separator $n^{O(\sqrt{\text{opt}(P)})}$.) \square

Our sub-exponential time algorithm in Theorem 4 is the first one for the covering problem of Hochbaum and Mass [9]. In addition the time complexity of our PTAS drastically improves time complexity of the original algorithm that was $n^{O(\frac{1}{\epsilon^2})}$ in [9].

Let \prec be a partial order on a finite set S . The dimension of \prec , denoted by $\dim(\prec)$, is the minimum number of total orders on S whose intersection is \prec [12]. We finish this section by stating a simple theorem that establishes some connections between partial orders, the dimension of interval orders and the concept of length introduced here.

Theorem 5 *Let G be a graph.*

(i) *If $l(G, G) = 1$, then G is an incomparability graph.*

(ii) *If G is an interval graph whose underlying interval order is \prec , then $\dim(\prec) \leq l(G, G) + 2$.*

Justification. For (i), let (C_1, C_2, \dots, C_k) be a clique cover of G so that for $x \in C_i$ and $y \in C_j$, we have $xy \in E(G)$ only if $|i - j| \leq 1$. Now for any $x, y \in V(G)$ with $x \in C_i$, $y \in C_j$ with $j > i$, define $x \prec y$, if and only, if $xy \notin E(G)$. It is easily seen that \prec is partial order on $V(G)$ so that \bar{G} or the

complement of G is a comparability graph, and hence G is an incomparability graph. We omit the proof of (ii). \square

Remarks: A proof of Theorem 5 (ii) has appeared in Congressus Numerantium 205 (2010), 105-111. Additionally, a small number of minor corrections were made to this paper after its publications in EuroCG2010.

References

- [1] Agarwal, P.K., Kreveld, M., Suri, S., Label placement by maximum independent sets in rectangles, Comput. Geometry: Theory and Appl. 11(3-4) 209-218, 1998.
- [2] Bodlaender H.L, A Tourist Guide through Treewidth. Acta Cybern. 11(1-2) 1-22, 1993.
- [3] Chan T., Polynomial-time approximation schemes for packing and piercing fat objects , Journal of Algorithms, 46(2), 178 - 189, 2003.
- [4] Chan T., Mahmood A., Approximating the piercing number for unit-height rectangles. CCCG 2005, 15-18, 2005.
- [5] Erlebach T., Jansen K. and Seidel E., Polynomial-time approximation schemes for geometric graphs. Proc. 12th ACM-SIAM Symposium on Discrete Algorithms (SODA'01), 671-679, 2001.
- [6] Fox J., Pach J., A separator theorem for string graphs and its applications, Combinatorics, Probability and Computing, 2009.
- [7] Fox J., Pach J., Separator theorems and Turn-type results for planar intersection graphs, Advances in Mathematics 219, 1070-1080, 2008.
- [8] Harry B. Hunt III, Marathe M. V., Radhakrishnan V., Ravi S. S, Rosenkrantz D. J., and Stearns R. E.. A Unified Approach to Approximation Schemes for NP- and PSPACE-Hard Problems for Geometric Graphs. Journal of Algorithms, 26, 135-149, 1996.
- [9] Hochbaum D.S., Maass W., Approximation Schemes for Covering and Packing Problems in Image Processing and VLSI. Journal of the Association for Computing Machinery, 32(1), 130-136, 1985.
- [10] Lipton, R.J., Tarjan, R.E., Applications of a planar separator theorem, SIAM J. Comput., 9(3), 615-628, 1980.

- [11] Miller, G.L., Teng, S., Thurston W., and Vavasis, S. A., Separators for Sphere-Packings and Nearest Neighborhood graphs, JACM, 44(1), 1-29, 1997.
- [12] Trotter, W.T., New perspectives on Interval Orders and Interval Graphs, Surveys in combinatorics, 1997.